Quantum Channel Negativity as a Measure of System-Bath Coupling and Correlation

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Complete positivity is a ubiquitous assumption in the study of quantum systems interacting with the environment, but the lack of complete positivity of a quantum evolution (called the "negativity") can be used as a measure of the system-bath coupling and correlation. The negativity can be computed from the Choi representation of a channel, is always defined and bounded, and can be used to understand environmentally induced noise in a quantum system.

I. INTRODUCTION

Complete positivity has become an ingrained part of the modern study of open quantum systems, but dynamics of quantum systems need not be completely positive [1–6] (and references therein). A non-completely positive quantum evolution is called "negative", simply because it is shorter to write than "non-completely positive". A composite quantum system is a quantum system under the control of the experimenter (called the "reduced system") along with the other quantum systems inaccessible to the experimenter that may still influence the dynamics of the reduced system (called the "bath", "environment", "reservoir", etc). It will be shown that a measurement of the negativity (defined below) will give an experimenter some understanding of the coupling and correlations between the reduced system and bath.

The negativity defined here is not the entanglement measure introduced by Vidal and Werner [7] with which it shares the name "negativity". The two measures share some mathematical similarities, including finding the sum of the negative eigenvalues of a given matrix using the trace norm, but the Vidal negativity is a function of the state and the channel negativity introduced here is a function of the reduced system dynamics. The channel negativity is not defined in terms of any kind of entanglement in the state.

A quantum channel ε is defined in the open systems setting as

$$\varepsilon(\rho) = \mathrm{Tr}_B \left(U \rho^\sharp U^\dagger \right) \quad , \tag{1}$$

where ρ is the initial state of the reduced system, U is the unitary evolution of the composite system, and $\sharp$ is called the "assignment map" (or "sharp operator"). The state ρ resides in the Hilbert space accessible to the experimenter in the lab, $\mathcal{H}^S$; i.e. $\rho \in \mathcal{H}^S$, and the evolution of the reduced system is found by "tracing out" the bath from the joint evolution of the reduced system and the bath [8]; i.e. $U \in \mathcal{H}^{SB} = \mathcal{H}^S \otimes \mathcal{H}^B$ where $\mathcal{H}^B$ is the Hilbert space of the bath. The partial trace operation, Tr_B, is an operator that allows expectation values of observables in the reduced system to be consistent with trivial extensions into a higher dimensional Hilbert space [9]. The sharp operation (or "assignment map") is an operation that injects the initial state of the reduced system into the higher dimensional Hilbert space of the composite system (i.e. $\rho^\sharp \in \mathcal{H}^{SB}$) [1, 3, 10] and is only defined on a subset of states in $\mathcal{H}^S$. The channel should take valid quantum states to valid quantum states, hence ε is to be positive (on some domain of states called the "positivity domain"), hermiticity-preserving, and consistent, i.e. $\mathrm{Tr}_B \left(\rho^\sharp \right) = \rho$. It is assumed here that $\mathcal{H}^S$, $\mathcal{H}^B$, and $\mathcal{H}^{SB}$ are all finite dimensional. More information about the mathematical structure of quantum information channels in the open system settings can be found in [10–12].

Every channel ε will have a Choi representation $\mathbf{C}$ given as

$$\mathbf{C} = \sum_{ij} E_{ij} \otimes \varepsilon\left(E_{ij}\right) > 0 \quad , \tag{2}$$

where E_{ij} is a matrix with the same dimensions as the reduced system that has a 1 at the ijth position and 0 everywhere else. The matrix $\mathbf{C}$ is commonly called "Choi's matrix" [13]. For a single qubit channel, Choi's matrix takes a simple block form, i.e.

$$\mathbf{C} = \left(\begin{array}{c|c} \varepsilon\left(|0\rangle\langle 0|\right) & \varepsilon\left(|0\rangle\langle 1|\right) \\ \hline \varepsilon\left(|1\rangle\langle 0|\right) & \varepsilon\left(|1\rangle\langle 1|\right) \end{array} \right) \quad . \tag{3}$$

The assumed linearity of the channel allows the off-diagonal blocks to be found using single qubit process tomography [14];e.g.

$$|0\rangle\langle 1| = \vec{r} \cdot \vec{\tau} \quad , \tag{4}$$

with the complex vector $\vec{r} = \{(1-i)/2, 1, i, (1-i)/2\}$ and the vector of states (or "tomography vector") $\vec{\tau} = \{|0\rangle\langle 0|, |+\rangle\langle +|, |+_i\rangle\langle +_i|, |1\rangle\langle 1|\}$ given $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|+_i\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$. The linearity of ε implies

$$\varepsilon\left(|0\rangle\langle 1|\right) = \vec{r} \cdot \varepsilon\left(\vec{\tau}\right) \quad ; \tag{5}$$

i.e. the matrix $\mathbf{C}$ is defined by the tomographic characterization of a channel.

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II. DEFINITION

The channel negativity is defined as

$$\eta \equiv \frac{\sum_i |\lambda_i|}{\sum_j |\lambda_j|} = \frac{1}{2}\left(1 - \frac{\mathrm{Tr}\,(\mathbf{C})}{||\mathbf{C}||_1}\right) \quad , \tag{6}$$

where λ is an eigenvalue of $\mathbf{C}$, $\lambda_i < 0\ \forall i$, and $||\mathbf{C}||_1 \equiv \sqrt{\mathbf{C}^\dagger \mathbf{C}}$ is the trace norm of $\mathbf{C}$. Notice, $\sum_j |\lambda_j| = \mathrm{Tr}\,(\mathbf{C})$ if and only if the negativity is zero.

The denominator is the sum of the absolute values of all of the eigenvalues of $\mathbf{C}$ and the numerator is the sum of the absolute values of all the negative eigenvalues of $\mathbf{C}$. Notice,

$$\mathbf{C} \geq 0 \rightarrow \sum_i |\Lambda_i| = 0 \rightarrow \eta = 0 \quad , \tag{7}$$

and if every eigenvalue were negative, then

$$\sum_i |\Lambda_i| = \sum_j |\Lambda_j| \rightarrow \eta = 1 \quad ; \tag{8}$$

however, the trace condition of $\mathbf{C}$ also implies

$$\mathrm{Tr}(\mathbf{C}) > 0 \Rightarrow \max(\eta) < 1 \quad ; \tag{9}$$

at least one eigenvalue of $\mathbf{C}$ will be non-negative because the trace of $\mathbf{C}$ is positive [13]. So, by construction,

$$\sup(\eta) = 1 \quad . \tag{10}$$

Also notice,

$$\sum_j |\Lambda_j| \geq \mathrm{Tr}\,(\mathbf{C}) = N \quad , \tag{11}$$

where $\mathbf{C}$ is a N^2xN^2 matrix. Hence, the channel negativity η is never undefined and $\eta \in [0, 1)$ with the vanishing negativity occurring if and only if the channel is completely positive.

The additivity of the channel negativity is still an open question because the composition of negative channels is not fully understood [11]. It is not clear that the channel negativity of a composite channel can be easily understood in terms of the negativity of its constituent channels. As such, this article will not discuss composite channels.

III. EXAMPLES

Before the discussion turns to the utility of the channel negativity, consider a few simple example calculations. The negativity requires a process tomography experiment to find the Choi representation of the channel, and in turn, the channel definition requires a sharp operation defined on the tomography vector of states used in the process tomography experiment. The sharp operations defined in the examples are given without motivation but discussion about physical realizations of different sharp operators (including the ones used in this article) can be found in [11, 12].

A. Root Swap Gate

Consider an example universe of two qubits. One qubit will be the reduced system and the other will act as the bath. If the composite dynamics are defined as the root swap gate

$$U_{\sqrt{Sw}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \quad , \tag{12}$$

then the channel becomes

$$\varepsilon(\rho) = \mathrm{Tr}_B \left(U_{\sqrt{Sw}} \rho^\sharp U^\dagger_{\sqrt{Sw}} \right) \quad . \tag{13}$$

Define the sharp operation on the canonical tomography vector $\vec{\tau}$ (introduced above) as

$$\rho^\sharp = \rho \otimes \left(H \rho H^\dagger \right) \tag{14}$$

where H is the Hadamard gate [14]. Process tomography of this channel yields a Choi representation of

$$\mathbf{C}_{\sqrt{Sw}} = \begin{pmatrix} \frac{3}{4} & -\frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{2}} & -\frac{1}{4} \\ \frac{1}{4} & \frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{2}} & \frac{1}{4} & -\frac{i}{2\sqrt{2}} \\ \frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{2}} & -\frac{1}{4} & \frac{i}{2\sqrt{2}} & \frac{3}{4} \end{pmatrix} \quad , \tag{15}$$

and a channel negativity of $\eta_{\sqrt{Sw}} \approx 0.149$.

B. CZ Gate

If the composite dynamics are defined as a controlled phase gate

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{16}$$

then the channel becomes

$$\varepsilon(\rho) = \mathrm{Tr}_B \left(CZ \rho^\sharp CZ^\dagger \right) \quad . \tag{17}$$

A Choi representation of this channel can be found using the sharp operator from the previous example; i.e.

$$\mathbf{C}_{CZ} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} - \frac{i}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} + \frac{i}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} + \frac{i}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad . \tag{18}$$

The negativity of this channel is $\eta_{CZ} \approx 0.167$.

In both of these examples, the channel negativity is fixed because the composite dynamics (i.e. the “coupling”) and sharp operations (i.e. the “correlation”) are fixed. The utility of the negativity comes from understanding the system-bath correlation and coupling when only one (or neither) of these things is fixed.

IV. PROBING THE BATH

A. Coupling Alone

Consider again a simple two qubit universe. The composite dynamics will be described by the general two qubit rotation

$$U_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \quad (19)$$

The sharp operation will be defined on the canonical tomography vector and will take the same form as before (i.e. $\rho^\sharp = \rho \otimes \left(H\rho H^\dagger\right)$), which leads to a Choi representation $\mathbf{C}_\theta$ of the channel $\varepsilon(\rho) = \mathrm{Tr}_B\left(U_\theta \rho^\sharp U_\theta^\dagger\right)$.

Unfortunately, it is difficult to find the spectrum of $\mathbf{C}_\theta$ in general. Notice, $\theta = 0$ and $\theta = 2\pi$ lead to $U_\theta = I$ where I is the two qubit identity operator. Define, η_θ to be the negativity of the channel represented by $\mathbf{C}_\theta$. If $\theta \in [0, 2\pi]$, then there are three points where $\eta_\theta = 0$: $\theta = 0$, $\theta = 2\pi$, and $\theta = \pi$. The negativity can be plotted as function of θ (see Fig. 1) to reveal a maximum negativity of $\eta_\theta \approx 0.24$ at the point $\theta = \pi/3$ and $\theta = 4\pi/3$. The negativity η_θ is a function of θ and can, in principle, be used to gain information about θ.

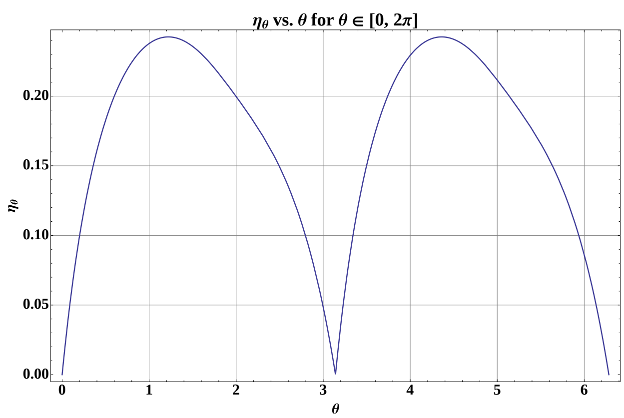

FIG. 1. The negativity η_θ can be plotted as a function of θ to show the dependency of the negativity on U_θ, despite the analytical difficulties of finding the spectrum of C_θ. See the text for definitions of U_θ and C_θ.

The negativity can be measured and if the above theoretical definition of the channel is assumed to be true, then θ can simply be read off Fig. 1. The negativity η_θ is measured in a single qubit tomography experiment, but U_θ can not be directly measured in any such experiment because the composite system contains a qubit defined to be beyond the reach of the experimenter (i.e. the bath qubit). Measurement of the negativity allows the experimenter to gain information about the environment that would otherwise be inaccessible.

B. Correlation Alone

The story with the correlation is very similar to the coupling. Suppose the composite dynamics are defined by the controlled phase gate, i.e. CZ, and the sharp operation is defined on the canonical tomography vector $\vec{\tau}$ as

$$\rho^{\sharp_\alpha} = \rho \otimes \left(U_\alpha \rho U_\alpha^\dagger\right) \quad (20)$$

with

$$U_\alpha = \alpha\sigma_1 + \sqrt{(1-\alpha^2)}\sigma_3 \ , \quad (21)$$

where $\sigma_1 = \sigma_x$ and $\sigma_3 = \sigma_z$ are the standard Pauli operators. Notice, U_α is unitary if $\alpha \in [0, 1]$ with $U_\alpha = H$ if $\alpha = 2^{-1/2}$.

The Choi representation of the channel $\varepsilon(\rho) = \mathrm{Tr}_B\left(CZ\rho^{\sharp_\alpha} CZ^\dagger\right)$ would be

$$\mathbf{C}_\alpha = \begin{pmatrix} 1 & 0 & 0 & \alpha\sqrt{1-\alpha^2} \\ 0 & 0 & \alpha\sqrt{1-\alpha^2} & 0 \\ 0 & \alpha\sqrt{1-\alpha^2} & 0 & 0 \\ \alpha\sqrt{1-\alpha^2} & 0 & 0 & 1 \end{pmatrix}$$

The spectrum of $\mathbf{C}_\alpha$ can be written down immediately as

$$\mathrm{spec}(\mathbf{C}_\alpha) = \{1 - x_\alpha, -x_\alpha, x_\alpha, 1 + x_\alpha\}$$

where $x_\alpha = \alpha\sqrt{1-\alpha^2}$. The negativity of this channel η_α can be bounded when $\alpha \in [0, 1]$ as

$$\alpha \in [0, 1] \Rightarrow \eta_\alpha \in \left[0, \frac{1}{6}\right] \ ,$$

with $\eta_\alpha = 0$ if $\alpha = 0$ or $\alpha = 1$ and $\eta_\alpha = 1/6$ if $\alpha = 2^{-1/2}$. The negativity η_α was already calculated for the case when $U_\alpha = H$ (i.e. $\alpha = 2^{-1/2}$) in the example calculations of the channel negativity.

The dependence of η_α on α can plotted to illustrate this idea a little more clearly (see Fig. 2). Again, the negativity η_α can be measured and if the above theoretical definition of the channel is assumed to be true, then α can simply be read off Fig. 2. In this example, as in the previous one, measurement of the negativity in a tomography experiment grants the experimenter knowledge about channel parameters that can not be measured directly.

C. Coupling and Correlation Together

These examples highlight a powerful usefulness of the negativity: measurement of the channel negativity yields information about the bath because the bath influences the channel through the composite dynamics

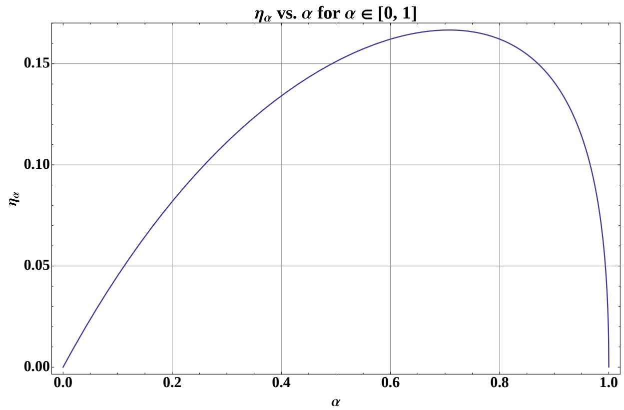

FIG. 2. The negativity η_α can be plotted as a function of α to show the dependency of the negativity on U_α. This example, like the example plotted in Fig. 1, illustrates how the negativity yields information about parameters in the channel definition.

and sharp operation. Unfortunately, the above examples are artificial in the sense of comparing the experimentally measured negativity to some known analytical definition of the channel (i.e. $\mathbf{C}_\theta$ and $\mathbf{C}_\alpha$). Typically, the experimenter will not have very detailed expectations about the form of the channel in the tomography experiment. Some assumptions might be made about the form of the sharp operation or composite dynamics, but it is rare to have a model of the channel complete enough (or which the experimenter has enough confidence in) to do the type of direct comparison between theory and experiment described in the examples. Typically, the experimenter would be doing tomography experiments precisely to figure out which assumptions about the sharp operation and composite dynamics are reasonable. Nevertheless, the main idea is sound. Even without precise, confidence-worthy models of the experimental channels, measurement of the negativity will provide information about the composite system.

Consider a slightly more complicated example with the composite dynamics given by U_θ and the sharp operation from the above example, i.e. consider the channel

$$\varepsilon(\rho) = \mathrm{Tr}_B \left(U_\theta \rho^{\sharp_\alpha} U_\theta^\dagger \right) \quad . \tag{22}$$

This single qubit channel combines the two above examples and will yield a negativity dependent on both the "correlation" (i.e. α) and the "coupling" (i.e. θ). The spectrum of $\mathbf{C}_\theta$ could not be written down for the general case, and the Choi representation of this channel $\mathbf{C}_{\theta\alpha}$ is likewise complicated. Notice $\theta = 0$ and $\theta = 2\pi$ yield

$$\mathbf{C}_{0\alpha} = \mathbf{C}_{2\pi\alpha} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

and $\theta = \pi$ yields

$$\mathbf{C}_{\pi\alpha} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad .$$

The Choi representations $\mathbf{C}_{0\alpha}$, $\mathbf{C}_{2\pi\alpha}$, and $\mathbf{C}_{\pi\alpha}$ are diagonally dominant and therefore, represent channels with vanishing negativities independent of the value of α [11].

The negativity of the channel represented by $\mathbf{C}_{\theta\alpha}$ can be plotted as a function of the full two parameter space (see Fig. 3). The maximum negativity $\eta_{\theta\alpha} \approx 0.24$ is

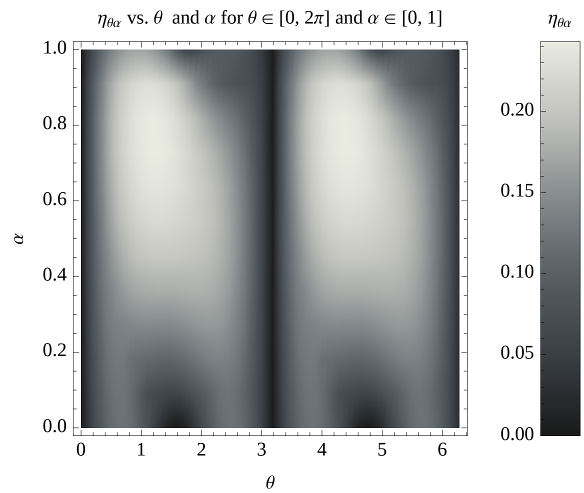

FIG. 3. The negativity $\eta_{\theta\alpha}$ can be plotted as a function of θ and α to show the dependency of the negativity on both the correlation and coupling in the channel.

achieved at $\alpha = 2^{-1/2}$ with either $\theta = \pi/3$ or $\theta = 4\pi/3$. Fig. 3 shows the same symmetry about $\theta = \pi$ as Fig. 1, and the maximum of Fig. 3 is at the individual maximums found in the single parameter spaces of Fig. 2 and Fig. 1.

The measured value $\eta_{\theta\alpha}$ can not uniquely identify a location in the two dimensional parameter space plotted in Fig. 3, and this inability is precisely the frustrating limitation of the bath information hidden in the negativity value. The correlation and coupling are confounded in such a way as to limit the experimenter's ability to gain precise information about either one from a measurement of the negativity. But, the measured negativity will limit the possible values of θ and α to some subset of the total parameter space which may be substantially smaller and might be helpful to the experimenter.

The negativity is subject to this confounding problem by definition, but it is also still useful beyond simply letting an experimenter know when she can not apply her completely positive theories (although that use is not without some worth). For instance, consider the use of the channel negativity as a kind of process fidelity.

D. Channel Negativity as a Process Fidelity

Consider a typical quantum information experiment: an experimenter is trying to implement a quantum gate in some new way and would like a measure of the "correctness" of the implemented gate. Suppose the gate to be implemented is the familiar controlled-phase gate CZ. If the experimenter actually implements

$$CZ' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-i\delta} \end{pmatrix} \quad , \tag{23}$$

where $\delta \in \mathbb{R} \geq 0$, then she would want to know how "close" CZ' is to CZ. There are numerous "gate fidelity" measures [15] [14], but a simple choice is the trace distance (derived from the trace norm) given as [15]

$$||M - N||_1 = \mathrm{Tr}\left(\sqrt{(M-N)^\dagger (M-N)}\right) \tag{24}$$

where M and N are any two operators. The trace distance is commonly used as a measure of distinguishability between two density matrices [15], but it can be used as a distance measure between CZ and CZ'. That trace distance is

$$||CZ - CZ'||_1 = 2\left|\cos\left(\frac{\delta}{2}\right)\right| \quad . \tag{25}$$

This form is very nice, but notice that a measurement of this trace distance would require a two qubit tomography experiment. The gate actually implemented in the experiment (i.e. CZ') would need to be characterized in an experiment which involves a tomography vector of 16 states.

Now consider a negativity measurement on a single qubit channel defined as

$$\varepsilon'(\rho) = \mathrm{Tr}_B\left(CZ'\rho^\sharp CZ'^\dagger\right) \quad , \tag{26}$$

where $\rho^\sharp = \rho \otimes \left(H\rho H^\dagger\right)$ is defined on the canonical single qubit tomography vector. The desired channel will have an expected negativity of $\eta_{CZ} \approx 0.167$ (see Eqn. 18).

The implemented channel will have a Choi representation

$$\mathbf{C}_{CZ'} = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{4}\left(3+e^{i\delta}\right) \\ 0 & 0 & \frac{1}{4} - \frac{e^{-i\delta}}{4} & 0 \\ 0 & \frac{1}{4}\left(1-e^{i\delta}\right) & 0 & 0 \\ \frac{1}{4}\left(3+e^{-i\delta}\right) & 0 & 0 & 1 \end{pmatrix} .$$

If $\delta = 0$ or $\delta = 2\pi$, then $CZ' = I$ where I is the two qubit identity operator and

$$\mathbf{C}_{CZ'} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

which has a vanishing negativity. If $\delta = \pi$, then $CZ' = CZ$ and the negativity of the implemented channel $\eta_{CZ'}$ will be equal to that of the expected channel, i.e. $\eta_{CZ'} = \eta_{CZ}$. Given $\delta \in [0, 2\pi]$, the point $\delta = \pi$ is the only point with this property. Hence, the quantity

$$\Delta = |\eta_{CZ} - \eta_{CZ'}| \quad ,$$

where Δ is a "negativity distance", can be used to determine how "far" the implemented gate CZ' is from the desired gate CZ. Fig. 4 shows the negativity of the implemented channel $\eta_{CZ'}$ as a function of the possible values of δ in CZ' given $\delta \in [0, 2\pi]$. The figure makes it

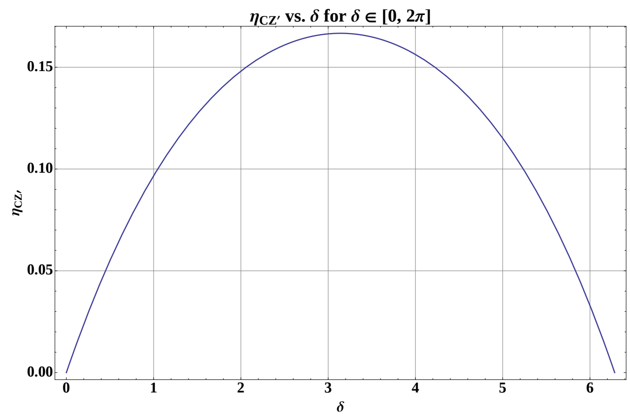

FIG. 4. The negativity $\eta_{CZ'}$ can be plotted as a function of δ to show the dependency of the negativity on CZ'. The negativity is maximized at $\delta = \pi$ and has minimums at $\delta = 0$ and $\delta = 2\pi$. See the text for discussions of these points and the use of this plot in determining Δ.

clear that

$$\Delta \in [0, \eta_{CZ}] \tag{27}$$

with

$$\Delta = 0 \Leftrightarrow \eta_{CZ'} = \eta_{CZ} \tag{28}$$

and $\Delta = \eta_{CZ}$ if $\delta = 0$ or 2π, but $0 < \Delta < \eta_{CZ}$ does not point to a unique δ.

It should be noted that the trace distance also suffers from the same inability to precisely determine δ, but the point of the trace distance or the negativity distance is not to provide a precise characterization of the implemented gate CZ'. The point is to provide a sense of distance of the implemented gate from the expected gate, and both the trace distance and the negativity distance provide this utility. Either distance measure could be used to determine "how far" the implemented gate is from the desired gate, but the experiment to measure $||CZ - CZ'||_1$ differs significantly from the experiment to determine Δ.

The negativity distance is still applicable given implemented gates depending on more than one parameter,

but the situation becomes expectedly more complicated. Consider an implemented gate

$$CZ'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-i\xi} & 0 \\ 0 & 0 & 0 & e^{-i\delta} \end{pmatrix} \quad , \tag{29}$$

which depends on two parameters $\{\delta, \xi\} \in \mathbb{R} \geq 0$ and will have some Choi representation $\mathbf{C}_{CZ''}$. Notice, $\delta = \xi$ yields

$$\mathbf{C}_{CZ''} = \begin{pmatrix} 1 & 0 & 0 & e^{i\delta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i\delta} & 0 & 0 & 1 \end{pmatrix} \tag{30}$$

which has a vanishing negativity. The negativity of this implemented channel $\eta_{CZ''}$ is plotted as a function of the two parameter space in Fig. 5 given $\{\delta, \xi\} \in [0, 2\pi]$.

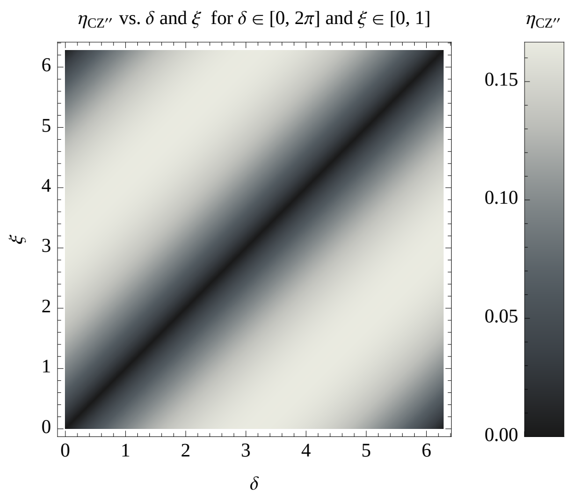

FIG. 5. The negativity $\eta_{CZ''}$ can be plotted as a function of both δ and ξ to show the dependency of the negativity on CZ''. See the text for discussions of how this two parameter space relates to the negativity distance introduced with the single parameter space of CZ' and Fig. 4.

This plot reveals the difficulty in using a negativity distance, e.g.

$$\Delta' = |\eta_{CZ''} - \eta_{CZ}| \tag{31}$$

to characterize the implemented gate CZ'' in the two parameter space of δ and ξ. Even a measurement of $\eta_{CZ''} = 0$ is not unambiguous, because $\delta = \xi \Rightarrow \eta_{CZ''} = 0$ but $\eta_{CZ''} = 0$ can mean $\delta = \xi$, $\delta = 0$ and $\xi = 2\pi$, or $\delta = 2\pi$ and $\xi = 0$. This problem will become more pronounced as the parameter space increases in dimension because there will be more combinations of parameters yielding vanishing (or not) negativities for the implemented channels.

The benefit of the negativity distance comes from the manner is which it would be measured. Complete characterization of the implemented process CZ' (or CZ'') requires a two qubit process tomography experiment. Such a complete characterization is required for the use of a distance measure akin to the trace distance. In contrast, the negativity distance between the implemented and expected channels is calculated using the measured negativity of a single qubit channel; i.e. only a single qubit process tomography experiment is needed. A tomography experiment requires a tomography vector which will have a length of N^2 where the number of states in the system $N = 2^n$ given n qubits. Hence, determining the negativity distance Δ requires a tomography vector with $2^{2\cdot 2} - 2^2 = 12$ fewer states than the trace distance $||CZ - CZ'||_1$.

The preparation and measurement of 4 states is a significantly simpler experiment then the preparation and measurement of 16 states. It is in this sense that the negativity distance is "less complex" than the trace distance, but notice that the negativity distance and trace distance experiments are not equal in all other aspects. The negativity measurement of the single qubit channel described above requires the implementation of a specific sharp operation. The two qubits input into CZ' need to be correlated in a very specific way which might be considered sufficiently complex enough to cause an experimenter to consider the full two qubit process tomography experiment "easier" to perform. The proposed sharp operation has already been given a proposed implementation procedure of projective measurements on a specifically entangled initial composite state [11, 12], and creating this entangled initial composite state might be very difficult in some experimental setups. This point is a fair criticism to the "simplicity" of the negativity distance measurements, but it is not always applicable. For example, it might be possible that the sharp operation could be implemented without significantly changing the experimental set-up used for the full two qubit process tomography experiment (see [11] for examples).

The negativity distance might lead to significantly simpler experiments in the characterization of gates of more than two qubits. Process tomography becomes much more difficult as the number of qubits increases due to the increased length of the tomography vector. In [11, 12], it was pointed out that negative channels can arise in some situations where the reduced system is correlated to only one of the bath qubits. Hence, a gate on more than two qubits might be characterized using a negativity distance measurement with a sharp operation between only two of the qubits. The sharp operation for such an experiment would be no more complex than it would be in an experiment to measure the Δ defined above, but the single qubit process tomography would be significantly simpler than the "greater than two qubit" process tomography that would otherwise be used to characterize the implemented gate.

V. CONCLUSIONS

The negativity provides information about the bath. The bath is defined by the experimenters ignorance. Hence, the negativity provides information about part of the quantum system traditionally considered unknowable. The inscrutable effects of the bath through the coupling and correlation can be observed through negativity measurements. Such information is academically interesting, but might have practical use in engineering quantum technologies and understanding the limitations of those technologies.

The negativity is a straightforward calculation using the Choi representation of a channel, which is part of all quantum process tomography experiments. These kinds of experiments are gaining popularity in the effort to engineer new quantum technologies, including quantum information processors. As an experimenter finds the superoperator describing the gate she is implementing, she can quickly calculate the channel negativity and gain some understanding of her system-bath coupling and correlation.

Such experiments have already measured non-zero negativities. Consider an experiment conducted by Cory et al. [16], in which process tomography is performed on a three-qubit NMR quantum information processor. The process tomography of this experiment leads to a "non-completely positive superoperator", i.e. a negative channel, which is characterized using a "positivity". The positivity ϱ is related to the channel negativity as

$$\eta = \frac{1-\varrho}{2-\varrho} \ . \tag{32}$$

The positivity of the experimental data in this NMR experiment (which was $\varrho = 0.60$) corresponds to a negativity of $\eta \approx 0.29$. These authors have experimental evidence of their system-bath coupling and correlation, and attempting to match this negativity measurement to theoretical models will lead to a better understanding of the open system dynamics of their experiment.

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